Ramanujan and the Partition Function

The story of Srinivasa Ramanujan, who wrote a letter to G. H. Hardy in early 1913 full of remarkable mathematical statements, came to Trinity in 1914 to work with Hardy, became a Fellow of the Royal Society and a Fellow of Trinity in 1918, and died in 1920 at the age of 32, is a famous one.

Ramanujan is now known as perhaps the purest mathematical genius there has ever been, and the body of work he left behind has had a deep influence on mathematics that continues to this day. Much of his work is too difficult to explain to the layperson, but one of his most notable results, concerning the so-called partition function, is an exception to this, and gives some idea of how unusual a mathematician he was.

To appreciate it fully, it helps to be aware of an important distinction. Consider first the well-known Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., where each term of the sequence is obtained by adding together the previous two terms. A natural question to ask is whether there is a formula for \( F_n \), the \( n \)th Fibonacci number, and it turns out that there is. Rather surprisingly (at least until one has seen how the formula is derived), \( F_n \) is equal to

\[
\frac{1}{\sqrt{5}} \left( \left( 1 + \sqrt{5} \right)^n - \left( 1 - \sqrt{5} \right)^n \right).
\]

This is an example of an exact result: it tells us that \( F_n \) is given exactly by an expression that is built up out of standard operations such as addition, multiplication, taking square roots, and raising to powers.

Now let us look at the sequence 2, 3, 5, 7, 11, 13, 17, 19, 23, ..., of prime numbers. Is there a formula for \( p_n \), the \( n \)th largest prime number? Many people have tried to find one in the past, but the overwhelming consensus is, and has been for a long time, that no useful formula for \( p_n \) exists. However, there is a simple and very useful approximate formula: the \( n \)th prime number is roughly equal to \( n \ln(n) \), where \( \ln \) is the natural logarithm. A better approximation turns out to be \( n \left( \ln(n) + \ln(\ln(n)) - 1 \right) \). This result, proved independently by Hadamard and de la Vallee Poussin at the end of the 19th century, is one of the most famous in all mathematics. However, it leaves open the question how good the approximation is, and this is the most famous unsolved problem in all mathematics, known as the Riemann hypothesis.

The partition function counts the number of ways a number can be split up as a sum of positive integers in decreasing order. To give an example, since the value of \( p(6) \) is 11. We can put the values together to form another sequence, which begins 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, ... .

The partition function counts the number of ways a number can be split up as a sum of smaller numbers. To give an example, here is a list of ways of splitting up the number 5.

\[
\begin{align*}
5 = & \quad 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
& \quad 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.
\end{align*}
\]

There are seven ways of doing it (if we include the “trivial” one that writes 5 as 5), so the value of the partition function at 5 is 7. We write this as \( p(5) = 7 \).

To be precise, \( p(n) \) is the number of ways of writing \( n \) as a sum of positive integers in decreasing order. To give another example, since

\[
\begin{align*}
6 = & \quad 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 3 \\
= & \quad 3 + 2 + 1 = 3 + 1 + 1 + 1 = 2 + 2 + 2 \\
= & \quad 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 \\
= & \quad 1 + 1 + 1 + 1 + 1 + 1.
\end{align*}
\]

the value of \( p(6) \) is 11. We can put the values together to form another sequence, which begins 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, ... .

We can of course try to find a formula for the \( n \)th term of this sequence, but it is better to start with a less precise question: does this seem like the kind of problem where one would expect an exact formula, or is it more likely to be one where the best we can hope for is a good approximation? The
evidence is mixed. On the one hand, there are several very interesting exact relationships between different functions related to the partition function, which leads one to think that perhaps an exact formula can be found. On the other hand, the sequence has some curious features that would be hard to capture with a neat formula: for instance, if we take the sequence of differences of successive terms, we obtain the sequence 1, 1, 2, 2, 4, 4, 7, 8, 12, 14, 21, 24, 34, 41, 55, 66, 88, . . . , which increases in a strange stuttering way.

Here is a paragraph from a famous paper, entitled *Asymptotic Formulae in Combinatory Analysis*, by Hardy and Ramanujan.

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities, many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of asymptotic formulae, one may fairly say, there are none. So true is this, in fact, that we have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of \( p(n) \).

Many mathematicians, myself included, would have been satisfied with this approximation. But Hardy and Ramanujan went much further, obtaining the following even better approximation for \( p(n) \):

\[
\frac{1}{2\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{e^{\pi \sqrt{n/3}}}{n^{1/4}} \exp \left( \frac{\pi \sqrt{2/3}}{n^{1/4}} \right)
\]

I shall not attempt to say what all the component parts of this formula mean. Suffice to say that the mathematics that goes into this formula is very deep (for instance, the appearance of the number 24 in the formula is related to the appearance of the same number in seemingly very different contexts).

How good is this approximation? Even Hardy and Ramanujan themselves were surprised by the answer. If you choose \( \nu \), the number of terms in the sum, in an appropriate way (it should be around the square root of \( n \)), then this formula gives an answer that is accurate to within less than 1. In other words, although this is an asymptotic formula, it is so accurate that you can work out the exact answer by simply working out the approximation and taking the nearest integer to it. Let me quote the paper again.

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulae of the theory of the linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behaviour of \( f(x) \) near any assigned point of the unit circle. It is to an appropriate use of these formulae that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

Hardy and Ramanujan were quite right about this: the accuracy was indeed found startling and still is today. Even this does not fully explain why their result is so important. Another reason is that it introduced into number theory a technique known as the circle method, which very roughly means studying sets of numbers with the help of Fourier analysis, and which has been used to solve many other notable problems. A recent example is a proof by Harald Helfgott that every odd number greater than 5 can be written as a sum of three prime numbers, solving the so-called ternary Goldbach problem. I shall leave the last word to Littlewood.

*We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him. Ramanujan’s genius did have this one opportunity worthy of it.*