Answers to booklet on proof

Claim B1: If you play the game of changing the word SHIP into the word DOCK, one letter at a time, and making a real word at each stage - like SHIP-SHOP-CHOP-COOP-LOOP-LOOK-LOCK-SOCK-DOCK - then at least one of the in-between words must have two vowels.

Proof: Let α be the first word in the game that does *not*, like SHIP, have a vowel as the third letter. [It's LOCK in the example above].

Let β be the word before α in the game. [It's LOOK in the example above]. Then β must have a vowel as the third letter.

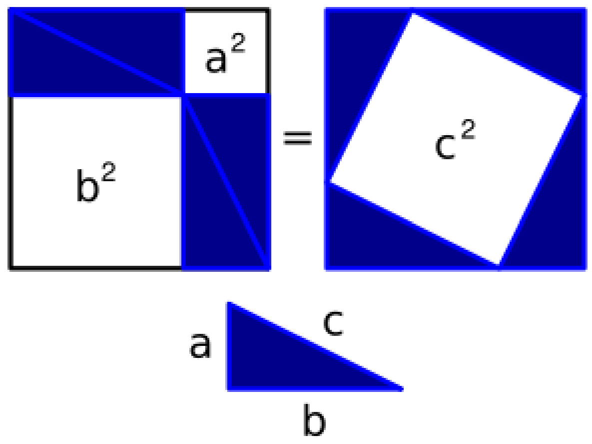
α must have a vowel in it *somewhere* because all words in English have at least one vowel. It can't be in the third place, so it must be in some other place: first, second, or fourth.

In the game we change only one letter at a time. Since β has a vowel as the third letter, and α has a consonant, it must be the third letter we change between β and α.

Therefore the vowel in α (not in the third place) is not changed between β and α, and must also be in β.

Therefore β has a vowel in the third place, and also a vowel not in the third place. Therefore β has two vowels. □

Claim B2: In a right-angled triangle, the square of the hypotenuse = the sum of the squares on the other two sides.



Proof: Let the hypotenuse be c, and the other two sides a and b. Draw a square with side c, and add a copy of the triangle to each side of the square (as in the right hand side of the picture), to create a bigger square.

Then move the triangles around within the big square to rearrange them as in the left-hand side of the picture.

The area of the bigger square not covered by the (shaded) triangles is the same before and after rearrangement.

Therefore $c^2 = a^2+b^2$ □

[There are hundreds of other proofs of this claim. You may find another one you prefer].

Claim B3: A line touching a circle is a tangent (i.e. touches it at only one point) if and only if it is at right angles to the radius at the point where it touches the circle.

Proof: Let F be the centre of the circle. First we prove that the line from F to any line ED which meets ED at right angles is shorter than any other line from F to the line ED. [Strictly speaking we should not assume that a line exists from F to ED which is at right angles to ED. But it can be proved that there is always one right-angle line of that sort, and only one].

Let C be the point where the right-angle line from F meets ED, and let G be any other point on ED. Then FCG is a right angled triangle with FG as hypotenuse, and so by claim B1 FG is greater than FC. Therefore FC is the shortest line from F to the line ED.

So, if C is on the circle, every other point on ED is outside the circle. Therefore ED is a tangent.

Now we prove the other part of the claim: that if ED is a tangent, meeting the circle at C, then FC is at right angles to ED.

If ED is a tangent, meeting the circle at C, then every other point on ED is outside the circle.

So the distance from F to every other point on ED is greater than FC.

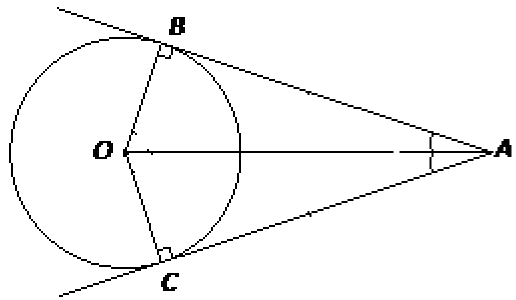
Therefore FC is the shortest line from F to ED. Therefore FC is at right angles to ED □

Claim B4: Two tangents drawn to the same circle from the same point have equal lengths.

Proof: By claim B3, the tangent AB is at 90 degrees to the radius OB, and the tangent AC is at right angles to OC

Therefore the triangles ABO and ACO are both right-angled, have a hypotenuse in common (OA), and have equal sides OB and OC

Therefore they are congruent. Therefore AB = AC □



Claim B5: For all $n \in \mathbb{Z}^+$, and all $x \neq 1$, $(1-x^n)/(1-x) = 1+x+x^2+x^3+\ldots+x^{n-1}$

Proof: $1\ (1+x+x^2+x^3+\ldots+x^{n-1}) = 1+x+x^2+x^3+\ldots+x^{n-1}$ **[1]**

$x\ (1+x+x^2+x^3+\ldots+x^{n-1}) = x+x^2+x^3+\ldots+x^{n-1}+x^n$ **[2]**

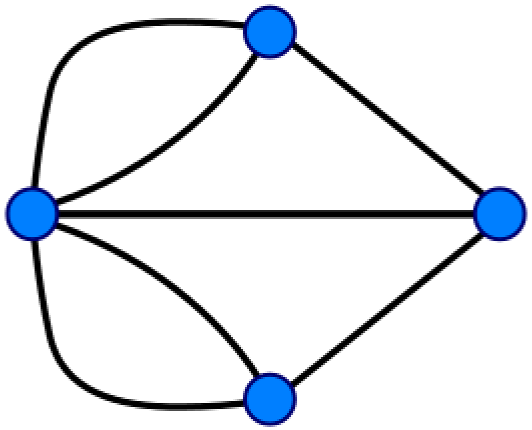
Subtract **[2]** from **[1]**

$(1-x)\ (1+x+x^2+x^3+\ldots+x^{n-1}) = (1-x^n)$

If $x \neq 1$, we can divide both sides by $(1-x)$, so

$(1-x^n)/(1-x) = 1+x+x^2+x^3+\ldots+x^{n-1}$ □

Claim B6: In the old German city of Königsberg, there were seven bridges joining four different areas of the city. It was impossible to walk round the city crossing each bridge exactly once.



Proof: Redraw the picture so that each area is representing by a vertex and each bridge by an edge.

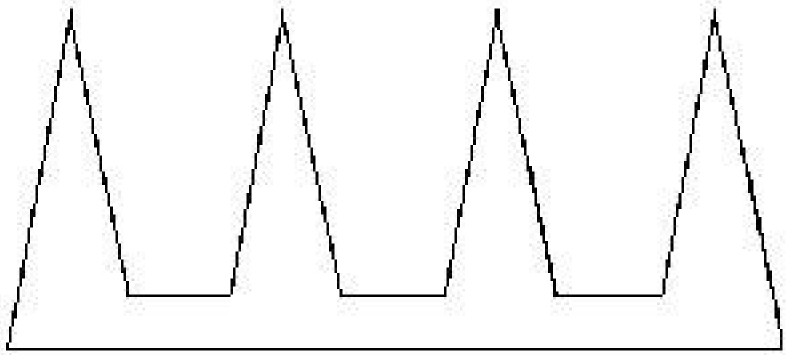
In any pattern like this, if a vertex (area) is neither the start point nor the end point, then a path round the pattern travelling each edge exactly once must enter by one edge and leave by another. The path "covers" edges joining to that vertex two at a time, not one at a time. If the path travels every edge exactly once, then that vertex must be "even" in the sense that an even number of edges join to it. So, if we have a path travelling every edge exactly once, then every vertex except possibly two (the start and the end) must be "even".

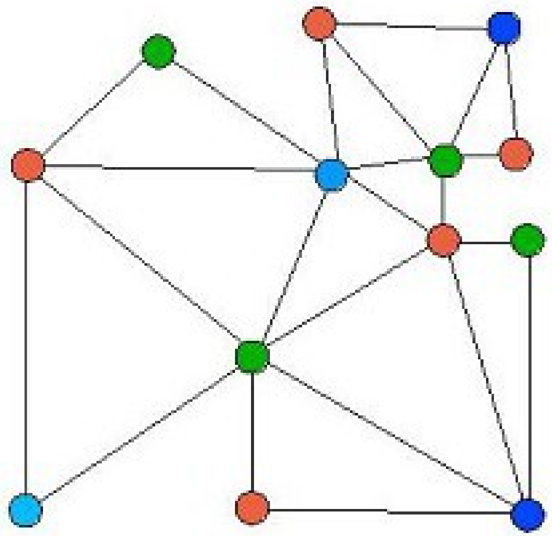
However, in Königsberg all the vertices were "odd" (an odd number of edges joined to them). So no path travelling every edge exactly once was possible □

[This proof, like some others, proves more than the original claim. It proves that any pattern with more than two odd vertices has no path travelling every edge exactly once].

Claim B7 (The Art Gallery Theorem): If an art gallery's floor plan is a polygon with n sides, then to ensure that every bit of the gallery is watched the gallery will need a maximum of ⌊n/3⌋ guards. (⌊n/3⌋ means the biggest whole number less than or equal to n/3. For example ⌊9/3⌋ = 3, ⌊10/3⌋ =3, ⌊11/3⌋ =3, ⌊12/3⌋ =4).

Proof: Of course if the floor plan is convex, just one guard in the middle will do. But we want to find a maximum number of guards which will be enough however pesky the floor plan.





First we show that for every n, there is a floor plan pesky enough that ⌊n/3⌋ guards are necessary. If the floor plan is as in the picture on the left, an extra guard is necessary for each triangular bit, and that makes ⌊n/3⌋ guards.

Now we will show that ⌊n/3⌋ guards is enough however pesky the floor plan. Choose two neighbouring corners of the polygon and colour them red and green. Starting from there, divide up the whole polygon into triangles step by step, adding a new polygon corner each time, with each triangle's corners being also corners of the polygon. (We should really prove it's possible to do that division into triangles. But it's a little tricky to give a strict proof, and if you experiment with a few polygons you will be convinced it is always possible).

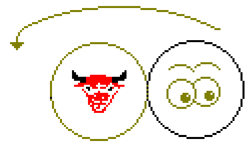
Remember the first two corners are coloured red and green. As you do the division into triangles, colour the first corner you add blue.

As you add each corner after that, colour it a different colour from the other two corners of the triangle of which it is part.

When you have finished the division into triangles, choose the least-used colour. It will have been used at most ⌊n/3⌋ times. Put a guard on each corner of the least-used colour. You then have a guard in every triangle. Every part of the polygon can be seen by the guard in the triangle of which it is part.

Therefore ⌊n/3⌋ guards are always enough □

Claim B8: If two identical coins of equal radius are placed side by side on a table, with one of them fixed, and both head up; and one is rotated one about the other, without slipping, until it is on the other side of the fixed coin; then the rotated coin ends head up.



Proof: Let r = radius of a coin

Then the centre of the rotated coin is distance 2r from the centre of the fixed coin

Therefore, when the rotated coin has reached the other side of the fixed coin, its centre has travelled a distance 2πr without slipping, the equivalent of a whole circumference

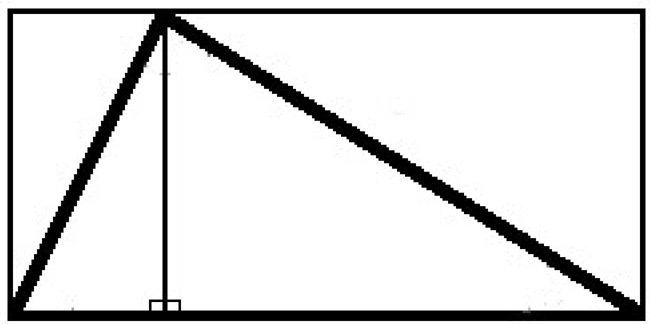
Therefore the rotated coin has rotated through 360 degrees

Therefore the rotated coin ends head up □

Claim A2: The area of a triangle = ½ × base × height

Proof: Since the angles of a triangle add to 180, only one angle at most in a triangle can be more than 90.

Draw the triangle so that both angles at the bottom are less than 90, and then draw a rectangle with the same base and the same height as the triangle. Then draw a vertical line from the top of the triangle to the base.

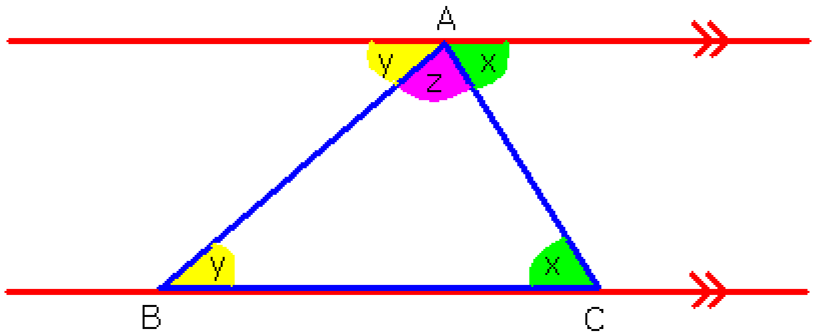


Then the two smaller triangles on the left have a corresponding side in common, and the same angles. Therefore they are congruent. Therefore they have the same area. Therefore the two smaller triangles on the right also have the same area as each other.

Area of the original triangle = area of one of the smaller triangles on the left + area of one of the smaller triangles on the right. Area of the rectangle = area of both smaller triangles on the left + area of both smaller triangles on the right. Therefore, area of triangle = half area of rectangle = ½ × base × height □

Claim A3: The angles of a triangle add up to 180

Proof: Draw a line parallel to the base through the top corner of the triangle, and label the angles as shown.



The two angles marked y are equal because they are alternate. The two angles marked x are equal because they are alternate.

At the top of the diagram we see x+y+z = a straight line = 180

But the angles inside the triangle also add up to x+y+z

Therefore the angles of the triangle add up to 180 □

[Note: Proving alternate angles are equal is a bit trickier, but you can assume it for this proof].

Claim A4. For any real coefficients b and c, r is a root of the equation $x^2+bx+c=0$ if and only if $(x–r)$ is a factor of $x^2+bx+c=0$

Proof: First we prove that if $(x–r)$ is a factor, then r is a root.

If $(x–r)$ is a factor, then $x^2+bx+c = (x–r)$.something

When $x=r$, the left-hand side of that equation = 0. Therefore the right-hand side = 0. Therefore r is a root.

Now prove that if r is a root, $(x–r)$ is a factor.

Do the long division of $(x–r)$ into x^2+bx+c. The remainder is a number. Call it d

Then $x^2+bx+c = (x–r)$.something + d for all x

Put $x=r$ in this equation. If r is a root, then the left hand side = 0. Therefore the right-hand side = 0

Therefore $d=0$. Therefore the remainder in the long division is zero, and $(x-r)$ is a factor □

Claim A5: If b, c, and d are real coefficients, and r_1, r_2, and r_3 are roots of the equation $z^3+bz^2+cz+d=0$, then

the sum of roots $r_1 + r_2 + r_3= –b$ and the product of roots $r_1r_2r_3=–d$

Proof: From claim A4, if r_1, r_2, and r_3 are roots, then $(z–r_1)$ and $(z–r_2)$ and $(z–r_3)$ are all factors

Therefore $z^3+bz^2+cz+d = (z–r_1)(z–r_2)(z–r_3)$

$= z^3 – (r_1 + r_2 + r_3)\, z^2$ + something. $z – r_1r_2r_3$

Therefore $r_1 + r_2 + r_3= –b$ and $r_1r_2r_3= –d$ □