Mark scheme for December 2016 FP2 mock

Question Number	Scheme	Marks
1 **(a)**	$\lvert z \rvert = 4$	B1
	$\arg z = \arctan\left(\frac{-2\sqrt{3}}{2}\right) = \arctan\left(-\sqrt{3}\right) = \frac{2\pi}{3}$ or 120°	M1A1 (3)
(b)	$z^6 = \left(4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)\right)^6 = 4^6\left(\cos 4\pi + \mathrm{i}\sin 4\pi\right)$ or $z^6 = \left(4\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^6$	M1
	$= 4096$ or 4^6 or 2^{12}	A1 cso (2)
	(a) and (b) can be marked together	
(c)	$z^{\frac{3}{4}} = 4^{\frac{3}{4}}\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)^{\frac{3}{4}} = 4^{\frac{3}{4}}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$	
	$w = \mathrm{i}2\sqrt{2}$ oe or any other correct root	B1
	$4^{\frac{3}{4}}\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + \mathrm{i}\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)^{\frac{3}{4}}$	M1
	$(n = 0$ see above$)$	
	$n = 1$ $w = 2\sqrt{2}$ oe	
	$n = 2$ $w = -\mathrm{i}2\sqrt{2}$ oe	
	$n = 3$ $w = -2\sqrt{2}$ oe	A1A1 (4)
		[9]

(a) B1 Correct modulus seen **Must** be 4

M1 Attempt arg using arctan, nos either way up. Must include minus sign or other consideration of quadrant. (arg = $\frac{\pi}{3}$ scores M0)

A1 $\frac{2\pi}{3}$ or 120^0 Correct answer only seen, award M1A1

(b) M1 apply de Moivre

A1cso 4096 or 4^6 Must have been obtained with the correct argument for z

(c) B1 For $w = \mathrm{i}2\sqrt{2}$ or any single correct root (0 or 0i may be included in all roots) in any Form including polar

M1 Applying de Moivre and use a correct method to attempt 2 or 3 further roots

A1A1 For the other roots (3 correct scores A1A1; 2 correct scores A1)

Accept eg $2\sqrt{2}, \sqrt{8}$, 2.83, $64^{\frac{1}{4}}$, $4^{\frac{3}{4}}$, $4096^{\frac{1}{8}}$ Decimals must be 3 sf min.

ALT 1 for (c):

$z^3 = 64 = w^4 \Rightarrow w = (\pm)2\sqrt{2}$ ($\pm$ not needed) B1

Use rotational symmetry to find other 2/3 roots M1

Remaining roots as above A1A1

ALT 2:

$z^4 = 64$ $z^2 = \pm 8$

$z = \pm 2\sqrt{2}$ $z = \pm\sqrt{-8} = \pm\mathrm{i}2\sqrt{2}$

B1 any one correct, M1 attempt remaining 2/3 roots; A1A1 as above

Question Number	Scheme	Marks
2 **(a)**	$(x+2)(x+3)^2 - 12(x+3) = 0$ OR $\dfrac{(x+3)(x+2)-12}{(x+3)} > 0$	M1
	$(x+3)(x^2+5x-6) = 0 \quad (x+3)(x+6)(x-1) = 0$	
	CVs: $-3, \ -6, \ 1$	B1,A1,A1
	$-6 < x < -3, \quad x > 1$ OR: $x \in (-6,-3) \cup (1,\infty)$	dM1A1 (6)
(b)	$x > 1$	B1 (1) [7]

(a)

M1 Mult through by $(x+3)^2$ and collect on one side or use any other valid method (NOT calculator)

Eg work from $\dfrac{(x+3)(x+2)-12}{(x+3)} > 0$

NB: Multiplying by $(x+3)$ is **not** a valid method unless the two cases $x > 3$ and $x < 3$ are considered separately or -3 stated to be a cv

B1 for -3 seen anywhere

A1A1 other cvs (A1A0 if only one correct)

dM1 obtaining inequalities using their critical values and no other numbers. Award if one correct inequality seen or any valid method eg sketch graph or number line seen

A1 correct inequalities and no extras. Use of … or ,, scores A0. May be written in set notation.

No marks for candidates who draw a sketch graph and follow with the cvs without any algebra shown. **Those who use some algebra** after their graph may gain marks as earned (possibly all)

(b) B1 correct answer only shown. Allow x ..1 if already penalised in (a)

Question Number	Scheme		Marks
3	$\frac{dy}{dx}+\frac{y}{\tan x}=3\cos 2x$		
	$\int \cot x\,dx=\ln\lvert\sin x\rvert$, IF $=\sin x$		M1
	$\sin x\frac{dy}{dx}+y\cos x=3\cos 2x\sin x$		
	$y\sin x=\int 3\cos 2x\sin x\,dx$		M1A1
	$y\sin x=\int 3\left(2\cos^2 x-1\right)\sin x\,dx$	$y\sin x=\frac{3}{2}\int\left(\sin 3x-\sin x\right)dx$	
	$y\sin x=3\left[-\frac{2}{3}\cos^3 x+\cos x\right](+c)$	$y\sin x=\frac{3}{2}\left[-\frac{1}{3}\cos 3x+\cos x\right](+c)$	dM1A1
	$y=\frac{3\cos x-2\cos^3 x+c'}{\sin x}$ oe	$y=\frac{-3\cos 3x+3\cos x+c'}{2\sin x}$	B1ft [6] (A1 on e-PEN)

M1 Divide by tan and attempt IF $e^{\int \cot x\,dx}$ or equivalent needed

M1 Multiply through by IF and integrate LHS

A1 correct so far

dM1 dep (on previous M mark) integrate RHS using double angle or factor formula

$$k\cos^2 x\sin x\to\pm\cos^3 x,\ k\sin^2 x\cos x\to k\sin^3 x,\ \cos 3x\to\pm\frac{1}{3}\sin 3x,\ \sin 3x\to\pm\frac{1}{3}\cos 3x$$

A1 All correct so far constant not needed

B1ft obtain answer in form $y =$ ... any equivalent form Constant must be included and dealt with correctly. Award if correctly obtained from the previous line

Alternatives for integrating the RHS:

(i) By parts: Needs 2 applications of parts or one application followed by a trig method. Give M1 only if method is complete and A1 for a correct result.

(ii) $y\sin x=\int 3\left(1-2\sin^2 x\right)\sin x\,dx=\int 3\sin x-6\sin^3 x\,dx$

Then use $\sin 3x=3\sin x-4\sin^3 x$ to get $y\sin x=\int\frac{3}{2}\left(\sin 3x-\sin x\right)dx$ and integration shown above - both steps needed for M1

	ALTERNATIVE: Mult through by cos x	
	$\sin x\frac{dy}{dx}+y\cos x=3\cos 2x\sin x$	M1
	$y\sin x=\int 3\cos 2x\sin x\,dx$	M1A1
	Rest as main scheme	

Question Number	Scheme	Marks
4 **(a)**	$r^2\left(r^2+2r+1\right)-\left(r^2-2r+1\right)r^2$	M1 A1
	$\equiv r^4+2r^3+r^2-r^4+2r^3-r^2$ or $r^2\left(r^2+2r+1-r^2+2r-1\right)$	
	$\equiv 4r^3$ *	A1 (3)
(b)	$\left(\sum_1^n 4r^3=\right)$ $\left(1\times 2^2-0\right)+\left(2^2\times 3^2-1^2\times 2^2\right)+\left(3^2\times 4^2-2^2\times 3^2\right)\ldots$	M1
	$+\left(n^2\times(n+1)^2-(n-1)^2\times n^2\right)$	
	$=n^2(n+1)^2$	A1
	$\sum_1^n r^3=\frac{1}{4}n^2(n+1)^2$	A1
	$\therefore \sum_1^n r^3=\left(\frac{1}{2}n(n+1)\right)^2=\left(\sum_1^n r\right)^2$	
	So $\left(1^3+2^3+3^3+....+n^3\right)=\left(1+2+3...+n\right)^2$ *	A1cso (4) [7] (B1 on e-PEN)

(a) M1 Multiply out brackets May remove common factor r^2 first

A1 a correct statement

A1 fully correct solution which must include at least one intermediate line

ALT: Use difference of 2 squares:

M1 remove common factor and apply diff of 2 squares to rest

A1 $r^2\left(r+1+r-1\right)\left(r+1-(r-1)\right)$

$=r^2\left(2r\times 2\right)$

A1 $=4r^3$

(b) M1 Use result to write out a list of terms; sufficient to show cancelling needed

Minimum 2 at start and 1 at end $\sum_1^n 4r^3$ or $\sum_1^n r^3$ need not be shown here or for next mark

A1 Correctly extracting $n^2(n+1)^2$ as the only remaining non-zero term.

A1 Obtaining $\sum_1^n r^3=\frac{1}{4}n^2(n+1)^2$

A1cso (Shown B1 on e-PEN) for deducing the required result.

Working from **either** side can gain full marks

Working from **both** sides can gain full marks provided the working joins correctly in the middle.

If ***r* used instead of *n***, penalise the final A mark.

Question Number	Scheme	Marks
5 (a)	$w = \dfrac{z}{z+3\mathrm{i}}$	
	$w(z+3\mathrm{i}) = z \quad z = \dfrac{3\mathrm{i}w}{1-w}$ or $\dfrac{-3\mathrm{i}w}{w-1}$	M1A1
	$\lvert z\rvert = 2 \quad \left\lvert \dfrac{3\mathrm{i}w}{1-w} \right\rvert = 2$	dM1
	$\lvert 3\mathrm{i}w\rvert = 2\lvert 1-w\rvert$	
	$w = u + \mathrm{i}v \quad 9(u^2+v^2) = 4\left((1-u)^2 + v^2\right)$	ddM1A1
	$9u^2 + 9v^2 = 4(1-2u+u^2+v^2)$	
(i)	$5u^2 + 5v^2 + 8u - 4 = 0$	dddM1
	$\left(u + \dfrac{4}{5}\right)^2 + v^2 = \dfrac{36}{25}$	
(ii)	So a circle, Centre $\left(-\dfrac{4}{5}, 0\right)$ Radius $\dfrac{6}{5}$ (oe fractions or decimals)	A1A1 (8)
(b)	Circle drawn on an Argand diagram in correct position ft their centre and radius	B1ft
	Region inside correct circle shaded no ft	B1 (2) **[10]**

(a) M1	re-arrange to $z = ...$
A1	correct result
dM1	dep (on first M1) using $\lvert z\rvert = 2$ with their previous result
ddM1	dep (on both previous M marks) use $w = u + \mathrm{i}v$ (or eg $w = x + \mathrm{i}y$) and find the moduli. Moduli to contain no is and must be +. Allow 9 or 3 and 4 or 2
A1	for a correct equation in u and v or any other pair of variables
dddM1	dep (on all previous M marks) re-arrange to the form of the equation of a circle (same coeffs for the squared terms
A1A1	deduce circle and give correct centre and radius. Completion of square may not be shown. Deduct 1 for each error or omission. (Enter A1A0 on e-PEN)
	Special Case: If $z = \dfrac{3\mathrm{i}w}{w-1}$ obtained, give M1A0 but all other marks can be awarded.
(b)	Mark diagram only - ignore any working shown.
B1ft	No numbers needed but circle must be in the correct region (or on the correct axis) for *their* centre and the centre and radius must be consistent (ie check how the circle crosses the axes) B0 if the equation in (a) is not an equation of a circle.
B1	Region inside the **correct** circle shaded. (no ft here)

Another way for Q.5

Leave blank

5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 3\text{i}}, \quad z \neq -3\text{i}$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C.

(a) (i) Show that C is a circle.

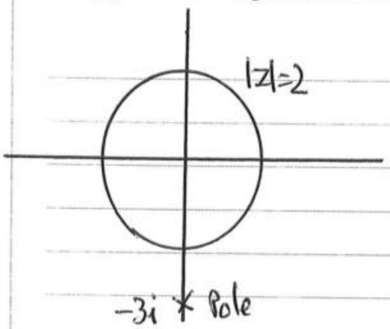
(ii) Find the centre and radius of C.

(8)

The region $|z| \leqslant 2$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) Shade the region R on an Argand diagram.

(2)



By the intersecting secants theorem, $z \mapsto \frac{1}{z^*}$ maps a circle into a circle when the pole $z=0$ is outside the circle.

Conjugation, translation, rotation, and enlargement all map circles to circles

∴ Möbius transforms map circles to circles when the pole is outside the circle.

∴ C is a circle.

By symmetry, the z-diameter in line with the pole maps to a w-diameter.

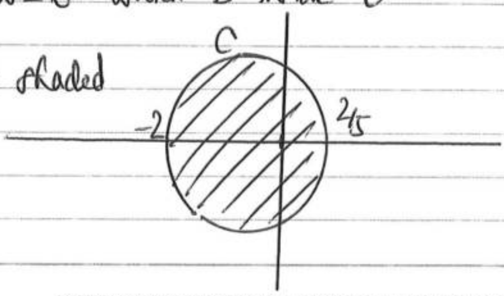
∴ $z = 2i$ and $z = -2i \mapsto w$ diameter-ends

∴ $\frac{2}{5}$ and -2 are w-diameter ends

∴ C has centre $-\frac{4}{5}$ and radius $\frac{6}{5}$.

$z = 0 \mapsto w = 0$ which is inside C

∴ R is as shaded



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Question Number	Scheme	Marks
6(a)	$z^n + z^{-n} = e^{in\theta} + e^{-in\theta}$	
	$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$	
	$= 2\cos n\theta$ *	M1A1
		(2)
(b)	$\left(z + z^{-1}\right)^5 = 32\cos^5\theta$	B1
	$\left(z + z^{-1}\right)^5 = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$	M1A1
	$32\cos^5\theta = \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right)$	
	$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$	M1
	$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$ *	A1
		(5)
(c)	$\cos 5\theta + 5\cos 3\theta + 10\cos\theta = -2\cos\theta$	M1
	$16\cos^5\theta = -2\cos\theta$	A1
	$2\cos\theta\left(8\cos^4\theta + 1\right) = 0$	
	$8\cos^4\theta + 1 = 0$ no solution	B1
	$\cos\theta = 0$	
	$\theta = \frac{\pi}{2},\ \frac{3\pi}{2}$	A1
		(4)
		[11]

Question Number	Scheme	Marks
7 (a)	$\frac{dy}{dx} = 2\tan x \sec^2 x$	B1
	OR $\frac{dy}{dx} = 2\tan x\left(1+\tan^2 x\right)$	
	$\frac{d^2y}{dx^2} = 2\sec^4 x + 4\tan^2 x \sec^2 x$ | $\frac{d^2y}{dx^2} = 2\sec^2 x + 2\times 3\tan^2 x\sec^2 x$	M1 A1
	$= 2\sec^4 x + 4\left(\sec^2 x - 1\right)\sec^2 x$ | $= 2\sec^2 x + 6\left(\sec^2 x - 1\right)\sec^2 x$	
	$= 6\sec^4 x - 4\sec^2 x$ *	A1cso (4)
(b)	$\frac{d^3y}{dx^3} = 24\sec^3 x \sec x \tan x - 8\sec^2 x \tan x$	M1A1
	$= 8\sec^2 x \tan x\left(3\sec^2 x - 1\right)$	A1cso (3)
(c)	$y_{\frac{\pi}{3}} = \left(\sqrt{3}\right)^2\ (=3)\quad \left(\frac{dy}{dx}\right)_{\frac{\pi}{3}} = 2\sqrt{3}\times\left(\frac{2}{1}\right)^2\ \left(=8\sqrt{3}\right)$	B1(both)
	$\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{3}} = 6\times 2^4 - 4\times 2^2 = 80$	
	$\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{3}} = 8\times 4\times\sqrt{3}\left(3\times 2^2 - 1\right) = 352\sqrt{3}$	M1(attempt both)
	$\tan^2 x = y_{\frac{\pi}{3}} + \left(x-\frac{\pi}{3}\right)\left(\frac{dy}{dx}\right)_{\frac{\pi}{3}} + \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{3}} + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{3}}$	
	$= 3 + 8\sqrt{3}\left(x-\frac{\pi}{3}\right) + 40\left(x-\frac{\pi}{3}\right)^2 + \frac{176}{3}\sqrt{3}\left(x-\frac{\pi}{3}\right)^3$	M1A1 (4)[11]

(a)B1 $\frac{dy}{dx} = 2\tan x \sec^2 x$

M1 attempting the second derivative, inc using the product rule or $\sec^2\theta = \tan^2\theta + 1$ **Must** start from the result given in (a)

A1 a correct second derivative in any form

A1cso for a correct result following completely correct working $\sec^2\theta = \tan^2\theta + 1$ must be d seen or used

Question Number	Scheme	Marks
8 (a)	$x = e^u \quad \frac{dx}{du} = e^u$ or $\frac{du}{dx} = e^{-u}$ or $\frac{dx}{du} = x$ or $\frac{du}{dx} = \frac{1}{x}$	B1
	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^{-u}\frac{dy}{du}$	M1
	$\frac{d^2y}{dx^2} = -e^{-u}\frac{du}{dx}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\frac{du}{dx} = e^{-2u}\left(-\frac{dy}{du} + \frac{d^2y}{du^2}\right)$	M1A1
	$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 2\ln x$	
	$e^{2u} \times e^{-2u}\left(-\frac{dy}{du} + \frac{d^2y}{du^2}\right) - 7e^u \times e^{-u}\frac{dy}{du} + 16y = 2\ln\left(e^u\right)$	dM1
	$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u \quad *$	A1cso (6)

(a) B1 for $\frac{dx}{du} = e^u$ oe as shown seen explicitly or used

M1 obtaining $\frac{dy}{dx}$ using chain rule here or seen later

M1 obtaining $\frac{d^2y}{dx^2}$ using product rule (penalise lack of chain rule by the A mark)

A1 a correct expression for $\frac{d^2y}{dx^2}$ any equivalent form

dM1 substituting in the equation to eliminate x **Only** u and y now Depends on the 2nd M mark

A1cso obtaining the given result from completely correct work

	ALTERNATIVE 1	
	$x = e^u \quad \frac{dx}{du} = e^u = x$	B1
	$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = x\frac{dy}{dx}$	M1
	$\frac{d^2y}{du^2} = 1\frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$	M1A1
	$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$	
	$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) - 7x \times \frac{1}{x}\frac{dy}{du} + 16y = 2\ln\left(e^u\right)$	
	$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u \quad *$	dM1A1cso (6)

B1 As above

M1 obtaining $\frac{dy}{du}$ using chain rule here or seen later

M1 obtaining $\frac{d^2y}{du^2}$ using product rule (penalise lack of chain rule by the A mark)

Question Number	Scheme	Marks
(b)	$m^2-8m+16=0$	
	$(m-4)^2=0 \quad m=4$	M1A1
	(CF =) $(A+Bu)e^{4u}$	A1
	PI: try $y=au+b$ (or $y=cu^2+au+b$ different derivatives, c = 0)	
	$\frac{dy}{du}=a \quad \frac{d^2y}{du^2}=0$	M1
	$0-8a+16(au+b)=2u$	
	$a=\frac{1}{8} \quad b=\frac{1}{16}$ oe (decimals must be 0.125 and 0.0625)	dM1A1
	$\therefore y=(A+Bu)e^{4u}+\frac{1}{8}u+\frac{1}{16}$	B1ft (7)
(c)	$y=(A+B\ln x)x^4+\frac{1}{8}\ln x+\frac{1}{16}$	B1 (1)
		[14]

(b) M1 writing down the correct aux equation and solving to m = … (usual rules)
A1 the correct solution ($m=4$)
A1 the correct CF – can use any (single) variable
M1 using an appropriate PI and finding $\frac{dy}{du}$ **and** $\frac{d^2y}{du^2}$ Use of $y=\lambda u$ scores M0
dM1 substitute in the equation to obtain values for the unknowns Dependent on the second M1
A1 correct unknowns two or three (c = 0)
B1ft a complete solution, follow through their CF and PI. Must have y = a function of u
Allow recovery of incorrect variables.

(c) B1 reverse the substitution to obtain a correct expression for y in terms of x No ft here
x^4 or $e^{4\ln x}$ allowed. Must start y = …