Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

6. The complex number $z = e^{i\theta}$, where θ is real.

(a) Use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

(2)

(b) Show that

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(5)

(c) Hence find all the solutions of

$$\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$$

in the interval $0 \leqslant \theta < 2\pi$

(4)

Similar questions: 47 and 61

47. (*a*) Given that $z = e^{i\theta}$, show that

$$z^n - \frac{1}{z^n} = 2i\sin n\theta,$$

where n is a positive integer.

(*b*) Show that

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

(*c*) Hence solve, in the interval $0 \leq \theta < 2\pi$,

$$\sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0.$$

61. (*a*) Use de Moivre's theorem to show that

$$\sin 5\theta = \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1).$$

(*b*) Hence, or otherwise, solve, for $0 \leq \theta < \pi$,

$$\sin 5\theta + \cos\theta\sin 2\theta = 0.$$

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

1 .

$$z = -2 + \left(2\sqrt{3}\right)\mathrm{i}$$

(a) Find the modulus and the argument of z.

(3)

Using de Moivre's theorem,

(b) find z^6, simplifying your answer,

(2)

(c) find the values of w such that $w^4 = z^3$, giving your answers in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$.

(4)

Similar questions: 52 and textbook p.69 Q.41

52. Solve the equation

$$z^5 = \mathrm{i},$$

giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$.

41 **a** Solve the equation

$$z^3 = 32 + 32\sqrt{3}\mathrm{i},$$

giving your answers in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, $-\pi < \theta \leqslant \pi$.

b Show that your solutions satisfy the equation

$$z^9 + 2^k = 0,$$

for an integer k, the value of which should be stated. *E*

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

3. Find, in the form $y = \mathrm{f}(x)$, the general solution of the differential equation

$$\tan x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 3\cos 2x \tan x, \qquad 0 < x < \frac{\pi}{2}$$

(6)

Similar questions 37 and 6

37. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y \cot 2x = \sin x, \qquad 0 < x < \frac{\pi}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

6. (*a*) Find the general solution of the differential equation

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x)y = \cos^3 x.$$

(6)

(*b*) Show that, for $0 \leq x \leq 2\pi$, there are two points on the x-axis through which all the solution curves for this differential equation pass. **(2)**

(*c*) Sketch the graph, for $0 \leq x \leq 2\pi$, of the particular solution for which $y = 0$ at $x = 0$. **(3)**

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

8. (a) Show that the transformation $x = e^u$ transforms the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 2 \ln x, \quad x > 0 \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2 y}{du^2} - 8 \frac{dy}{du} + 16y = 2u \qquad \text{(II)}$$

(6)

(b) Find the general solution of the differential equation (II), expressing y as a function of u.

(7)

(c) Hence obtain the general solution of the differential equation (I).

(1)

Similar questions 39 and 56

39. (*a*) Show that the transformation $y = xv$ transforms the equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + (2 + 9x^2)y = x^5, \qquad \text{I}$$

into the equation

$$\frac{d^2 v}{dx^2} + 9v = x^2, \qquad \text{II}$$

(*b*) Solve the differential equation II to find v as a function of x.

(*c*) Hence state the general solution of the differential equation I.

56. Given that $3x \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2 y}{dx^2} + 4y = k \cos 2x,$$

where k is a constant,

(*a*) calculate the value of k, **(4)**

(*b*) find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which at $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$. **(4)**

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

7.

$$y = \tan^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that $\frac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$

(4)

(b) Hence show that $\frac{d^3y}{dx^3} = 8\sec^2 x \tan x\,(A\sec^2 x + B)$, where A and B are constants to be found.

(3)

(c) Find the Taylor series expansion of $\tan^2 x$, in ascending powers of $\left(x - \frac{\pi}{3}\right)$, up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$

(4)

Similar question 33

33. Given that $y = \tan x$,

(*a*) find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

(3)

(*b*) Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

(3)

(*c*) Hence show that $\tan\frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$.

(2)

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

Not-quite-similar question from same general area: 53

53.

$$(1+2x)\frac{dy}{dx} = x + 4y^2.$$

(*a*) Show that

$$(1+2x)\frac{d^2y}{dx^2} = 1 + 2(4y-1)\frac{dy}{dx}. \qquad (1)$$

(2)

(*b*) Differentiate equation (1) with respect to x to obtain an equation involving

$\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, x and y.

(3)

Given that $y = \frac{1}{2}$ at $x = 0$,

(*c*) find a series solution for y, in ascending powers of x, up to and including the term in x^3.

(6)

Type of question: Ideas and formulas to use   What sort of answer you expect   Roadmap for solving the problem   How to check answer

2. (a) Use algebra to find the set of values of x for which

$$x + 2 > \frac{12}{x + 3}$$

(6)

(b) Hence, or otherwise, find the set of values of x for which

$$x + 2 > \frac{12}{|x + 3|}$$

(1)

Similar questions 5 and 18

5. Using algebra, find the set of values of x for which

$$2x - 5 > \frac{3}{x}.$$

18. Solve the inequality $\dfrac{1}{2x+1} > \dfrac{x}{3x-2}$.

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

4. (a) Show that

$$r^2(r+1)^2 - (r-1)^2 r^2 \equiv 4r^3$$

(3)

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

(4)

Similar questions 12 and 17

12. (*a*) Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

(*b*) Hence prove that $\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} \equiv \frac{n(5n+13)}{6(n+2)(n+3)}$.

17. (*a*) Express as a simplified fraction $\frac{1}{(r-1)^2} - \frac{1}{r^2}$.

(*b*) Prove, by the method of differences, that

$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}.$$

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 3\mathrm{i}}, \quad z \neq -3\mathrm{i}$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C.

(a) (i) Show that C is a circle.

(ii) Find the centre and radius of C.

(8)

The region $|z| \leqslant 2$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) Shade the region R on an Argand diagram.

(2)

Similar question 35

35. The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, \qquad z \neq -\mathrm{i}.$$

(*a*) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane. **(4)**

(*b*) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane. **(6)**

(*c*) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane. **(2)**

(*d*) Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane. **(2)**

Type of question:

Ideas and formulas to use

What sort of answer you expect

Roadmap for solving the problem

How to check answer

Not-quite-similar question from same area, 9

9. (*a*) The point P represents a complex number z in an Argand diagram. Given that

$$|z - 2i| = 2|z + i|,$$

(i) find a cartesian equation for the locus of P, simplifying your answer. **(2)**

(ii) sketch the locus of P. **(3)**

(*b*) A transformation T from the z-plane to the w-plane is a translation $-7 + 11i$ followed by an enlargement with centre the origin and scale factor 3.

Write down the transformation T in the form

$$w = az + b, \quad a, b \in \mathbb{C}.$$

(2)