Further Maths year 12-to-13 transition summer task 2017

City of London Academy Southwark

Please write your work on A4 lined paper, and hand it in at the start of the September term. These problems aim to:

- help bed in your knowledge of the method of differences
- help you retain the little about Maclaurin series we had time to do at the end of the summer term
- practise the sort of mathematical thinking which will be especially useful to you if you decide to try the extra exams, MAT or STEP, or if you just want to be a better mathematician generally

Because we had so little time on Maclaurin series, there are some reminder notes about it at the end of the task sheet.

1 Method of differences

1. You remember $\sum_1^n r = 1+2+3+4+\ldots+n = \frac{1}{2}n(n+1)$. Prove that formula using the method of differences, starting with

$$(r+1)r - r(r-1) = 2r$$

2. You remember $\sum_1^n r^2 = 1^2+2^2+3^2+4^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$. Prove that formula using your result from (1) and the method of differences, starting with

$$(r+2)(r+1)r - (r+1)r(r-1)$$

3. You remember $\sum_1^n r^3 = 1^3+2^3+3^3+4^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2$. Prove that formula using your results from (1) and (2) and the method of differences, starting with

$$(r+3)(r+2)(r+1)r - (r+2)(r+1)r(r-1)$$

4. What do you expect to be the highest power of n in a formula for $\sum_1^n r^4$? What do you expect to be the coefficient of that power in the formula?

5. Find a formula for $\sum_1^n r^4$ using your results from (1), (2), and (3) and the method of differences, starting with

$$(r+4)(r+3)(r+2)(r+1)r-(r+3)(r+2)(r+1)r(r-1)$$

6. Find A and B in the identity

$$\frac{1}{r(r+1)}=\frac{A}{r}+\frac{B}{(r+1)}$$

. Use your result and the method of differences to find $\sum_1^\infty \frac{1}{r(r+1)}$

2 MAT and STEP type problem-solving: introduction

In the next section are some multiple-choice questions from MAT papers. Write a choice for each question, *and a few words of explanation why you've made that choice.*

There are many different valid ways of doing most of these questions.

Some of these questions can be done by considering simple special cases.

Example: MAT 2011, B.

A rectangle has perimeter P and area A. The values P and A must satisfy: (a) $P^3 > A$, (b) $A^2 > 2P+1$, (c) $P^2 \geq 16A$, (d) $PA \geq A+P$.

Answer: Suppose it's a 1×1 square. Which of the inequalities is valid? Only (a) and (c). Suppose it is a very tiny square (say side 0.0001). Which of the inequalities is still valid? Only (c).

Some of them can be done by considering *symmetry*, or, speaking more broadly, *the shape* of the answer.

Example: MAT 2014. The inequality $x^4 < 8x^2+9$ is satisfied precisely when: (a) $-3 <$ x < 3; (b) $0 <$ x < 4; (c) $1 <$ x < 3; (d) $-1 <$ x < 9; (e) $-3 <$ x < -1.

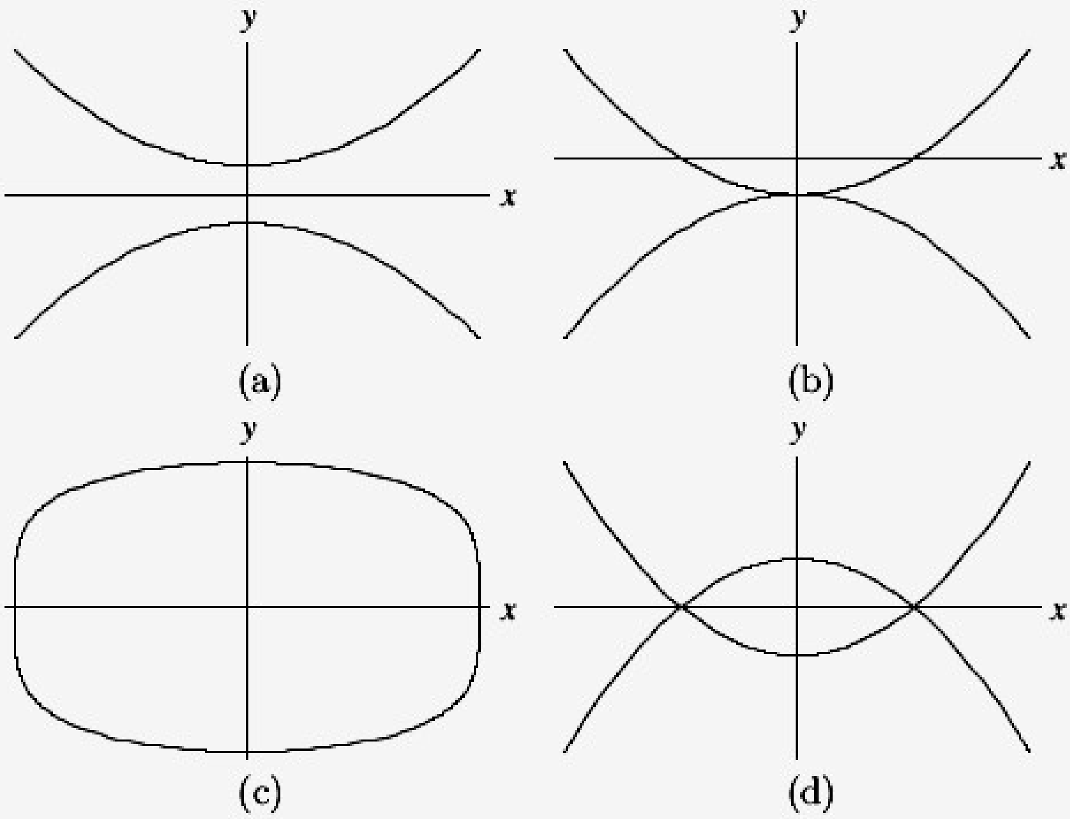
Answer: The inequality has only even powers of x in it, so it's true for x if it's true for $-$x. Therefore the interval for which it is satisfied must be: symmetrical around zero. Therefore the answer is: (a) $-3 <$ x < 3.

3 MAT and STEP type problem-solving: examples

Write a solution for the STEP question, and a choice for each multiple-choice question, *and a few words of explanation why you've made that choice.*

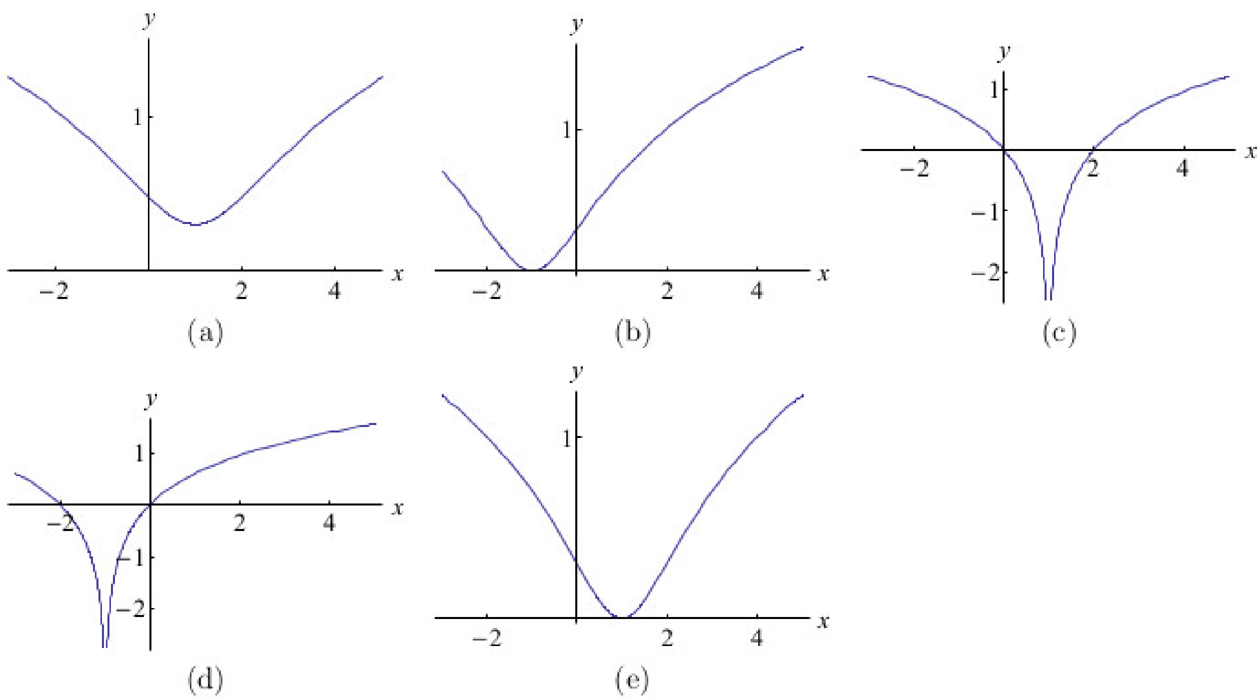
1. MAT 2013 d

D. Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



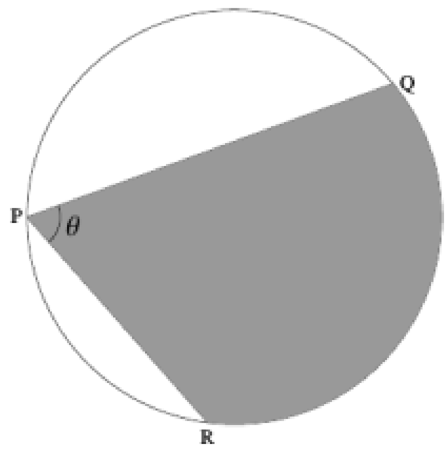
2. MAT 2014 b

B. The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



3. MAT 2012 j

J. If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,



then the largest possible area of the shaded region RPQ is

(a) $\theta\left(1+\cos\left(\frac{\theta}{2}\right)\right)$; (b) $\theta+\sin\theta$; (c) $\frac{\pi}{2}(1-\cos\theta)$; (d) θ.

4. MAT 2014 d

D. The reflection of the point $(1,0)$ in the line $y = mx$ has coordinates

(a) $\left(\frac{m^2+1}{m^2-1}, \frac{m}{m^2-1}\right)$, (b) $(1, m)$, (c) $(1-m, m)$,

(d) $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$, (e) $\left(1-m^2, m\right)$.

5. STEP 1999 Q.1

How many integers greater than or equal to zero and less than a million are not divisible by 2 or 5? What is the average value of these integers?

How many integers greater than or equal to zero and less than 4179 are not divisible by 3 or 7? What is the average value of these integers?

6. MAT 2014 f

The functions S and T are defined for real numbers by:

$S(x) = x + 1$; and $T(x) = -x$

The function S is applied s times and the function T is applied t times, in some order, to produce the function $F(x) = 8 - x$

It is possible to deduce that: (a) s = 8 and t = 1. (b) s is odd and t is even. (c) s is even and t is odd. (d) s and t are powers of 2. (e) none of the above.

7. MAT 2010 b

B. The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$

is

(a) $2^n + 1 - 2^{1-n}$, (b) $2^n + 2^{-n}$, (c) $2^{2n} - 2^{3-2n}$, (d) $\dfrac{2^n - 2^{-n}}{3}$.

8. MAT 2014 h

H. The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geqslant 2$,

$$\begin{aligned} F(n) &= F(n-1) + 2 && \text{if 2 divides } n \text{ but 3 does not divide } n; \\ F(n) &= F(n-1) + 3 && \text{if 3 divides } n \text{ but 2 does not divide } n; \\ F(n) &= F(n-1) + 4 && \text{if 2 and 3 both divide } n; \\ F(n) &= F(n-1) && \text{if neither 2 nor 3 divides } n. \end{aligned}$$

The value of $F(6000)$ equals

(a) 9827, (b) 10121, (c) 11000, (d) 12300, (e) 12352.

9. MAT 2014 j

J. For all real numbers x, the function $f(x)$ satisfies

$$6 + f(x) = 2f(-x) + 3x^2 \left(\int_{-1}^{1} f(t)\,\mathrm{d}t \right).$$

It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(a) 4, (b) 6, (c) 11, (d) $\frac{27}{2}$, (e) 23.

4 Maclaurin series problem A

See the reminder notes on Maclaurin series at the end of this tasksheet.

Find Maclaurin approximations

1. with powers up to x only
2. with powers up to x^2
3. with powers up to x^3

for the speed y of a particle thrown downwards at a starting speed of 2 ms^{-1}. The differential equation is $\frac{dy}{dx} = 10 - y$. In other words: air resistance = ym, and we're using g=10.

Using www.desmos.com or by hand, draw, in the same diagram, graphs of your three approximations for $x = 0$ to $x = 1.5$.

This particular differential equation can be solved exactly, and the solution is:

$$y = 10 - 8\mathrm{e}^{-x}$$

On the same diagram, draw that graph too.

Up to what value of x is the Maclaurin approximation using terms up to x^3 good?

For this problem, and the next one, either print out your graph from www.desmos.com and hand in the printed graph with your work; or screenshot your graph from www.desmos.com and email your screenshot; or hand in your graph done by hand. On www.desmos.com you get e^{-x} by entering $\exp(-x)$.

5 Maclaurin series problem B

Using the facts $\frac{d}{dx}(\sin x) = \cos x$, and $\frac{d}{dx}(\cos x) = -\sin x$, find Maclaurin approximations for $y = \sin x$:

1. with powers up to x only
2. with powers up to x^2
3. with powers up to x^3

Using www.desmos.com or by hand, draw, in the same diagram, graphs of your three approximations for for $x = 0$ to $x = 1.5$.

Draw the graph of $y = \sin x$ too.

Up to what value of x is the Maclaurin approximation using terms up to x^3 good?

6 Reminder notes on Maclaurin series

A differential equation is an equation which contains not just variables, like y and x, but also derivatives, like $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ or even $\frac{d^3y}{dx^3}$. Most differential equations can't be solved straightforwardly. Sometimes we can find a solution by better and better approximations looking like this:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

These approximations, called Maclaurin polynomials or Maclaurin series, get better and better as you add higher powers, terms in $x^4, x^5, x^6 \ldots$ or whatever.

We can also "code" *functions* by that sort of approximation

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

(How else would you, or at least, how would someone designing a calculator, work out $\ln 2$?)

Again, the approximations get better and better as you add higher powers, terms in $x^4, x^5, x^6 \ldots$ or whatever.

The term a_0 "anchors" the approximation to be right for y when x=0

The term $a_1 \cdot x$ "anchors" the approximation to be right for $\frac{dy}{dx}$ when x=0

The term $a_n \cdot x^n$ "anchors" it to be right for $\frac{d^ny}{dx^n}$, when x=0

The equations to "anchor" them correctly are:

$$a_n = \frac{1}{n!} \cdot \frac{d^n y}{dx^n}|_{x=0}$$

$n!$ is defined by: $n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$ By convention $0! = 1$

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

So

$$a_0 = y|_{x=0}$$

$$a_1 = \frac{dy}{dx}|_{x=0}$$

$$a_2 = \frac{1}{2!} \cdot \frac{d^2 y}{dx^2}|_{x=0}$$

At each stage, when you improve the approximation, all the a_n you have already worked out remain valid, and you need only work out the a_n for one bigger value of n.