Further Maths year 11-to-12 transition summer task 2017

City of London Academy Southwark

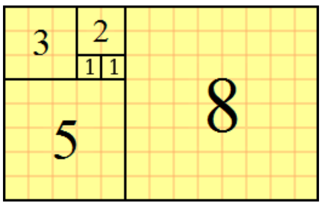
Please write your work on A4 lined paper, and hand it in at the start of the September term. These problems aim to introduce you to:

- being able to use imagination and trial-and-improvement to solve problems which need more than applying an known formula
- the idea of mathematical *proof* - clear, watertight arguments which prove a mathematical result *beyond doubt*
- "multiplication" with mathematical objects which are not numbers

1 Fibonacci numbers

The Fibonacci numbers are 1, 1, 2, 3, 5, 8 .... The rule for finding the nth Fibonacci number F_n is $F_n = F_{n-1} + F_{n-2}$

- Calculate the first twenty Fibonacci numbers

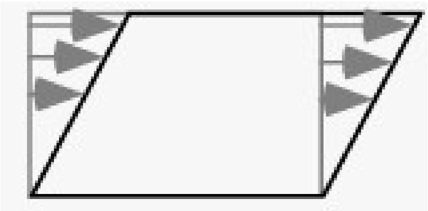


As in the diagram on the left, draw two squares of size 1 next to each other, then a square of side 2 adjoining those two, then a square of side 3 adjoining the 2-square and a 1-square... Continue your diagram of the Fibonacci numbers up to F_{10}.

- From your diagram, find an equation connecting the sum of the squares of the first n Fibonacci numbers, $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2$, with $F_n \cdot F_{n+1}$. Check that equation for $n = 1, 2, 3, 4, 5$
- S_n is defined as the sum of the first n Fibonacci numbers, so $S_3 = 1+1+2$, $S_4 = 1+1+2+3$, etc. Look for a general rule connecting S_n and F_{n+2}.
- *Prove* that rule, i.e. make a convincing argument that it keeps on being true however big n gets, a million, a billion, a billion billion, whatever.

2 Linear transformations

Linear transformations, as you will study them in Further Maths, include rotations, reflections, enlargements, as you studied them in Year 9. Not translations. They also include two other sorts of transformation which transform lines into lines, leave the origin where it is, and are can be described by linear equations. The two other sorts are shearing (left-hand diagram below); and scaling, i.e. enlarging by different amounts in different directions, e.g. stretching horizontally and squeezing vertically.



1. If the left-hand fairy is the original, which transformation - rotation? reflection? shearing? scaling? - does each other fairy show?

2. What sort of shape do linear transformations transform triangles into?

3. Do linear transformations transform squares into squares? circles into circles?

4. Define the multiplication $S \cdot T$ of the linear transformations S and T as doing T then S. Examples: rotating by 30° anticlockwise then rotating by 60° anticlockwise. Or rotating by 90° anticlockwise then stretching $2\times$ vertically and squeezing $\frac{1}{2}\times$ horizontally. Is $S \cdot T$ always a *linear* transformation (transforms lines to lines, leaves origin where it is, can be described by linear equations) if S and T are linear transformations?

5. For linear transformations S and T it is not *always* true that $S \cdot T = T \cdot S$. Find an example of S and T where $S \cdot T \neq T \cdot S$

6. Is it always true that if S, T, and U are linear transformations, $S \cdot (T \cdot U) = (S \cdot T) \cdot U$?

7. If we limit ourselves to the subgroup of linear transformations made up by enlargements and rotations - excluding reflections, unequal scalings, and shearings - now is it always true that $S \cdot T = T \cdot S$?

8. Draw diagrams to show (1) an enlargement by 1 and a rotation of zero (2) an enlargement by -1 and a rotation of zero (3) an enlargement by 1 and a rotation of 90° anticlockwise.

9. "Code" the number x as an enlargement by x and a rotation of zero. Can you find an enlargement-and-rotation transformation S for which $S^2 = 4$? And one for which $S^2 = -1$?