I-2009-4

The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta\,.$$

In the case $\alpha=2\beta$, show that $\cos\beta=\frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

The biggest angle α and the smallest angle β must be opposite the sides $p+q$ and the sides $p-q$ respectively; so:

$\cos\alpha=-\frac{(p+q)^2+(p-q)^2+p^2}{2p(p-q)}$

$\ldots=\frac{-4pq+p^2}{2p(p-q)}=\frac{p-4q}{2(p-q)}$

$\cos\beta=-\frac{(p-q)^2-(p+q)^2-p^2}{2p(p+q)}$

$\ldots=\frac{4pq+p^2}{2p(p+q)}=\frac{p+4q}{2(p+q)}$

$4(1-\cos\alpha)(1-\cos\beta)=\frac{(p+2q)(p-2q)}{p^2-q^2}$

$\cos\alpha+\cos\beta=\frac{(p-4q)(p+q)+(p+4q)(p-q)}{2(p^2-q^2)}=\frac{2p^2-8q^2}{2(p^2-q^2)}=\frac{(p+2q)(p-2q)}{p^2-q^2}$, as required.

If $\alpha=2\beta$, then if $c=\cos\beta$, $\cos\alpha=2c^2-1$, and so

$4(2-2c^2)(1-c)=2c^2-1+c$

$8c^3-10c^2-9c+9=0$

$0=(c+1)(8c^2-18c+9)=(c+1)(4c-3)(2c-3)$

so $c=\frac{3}{4}$, the other two solutions for c not giving triangles.

Put $q=1$, and $\frac{3}{4}=\frac{p+4}{2(p+1)}$, so $6p+6=4p+16$, $p=5$, and sides are 6:5:4.